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# Strong Unique Continuation Property of Two-dimensional Dirac Equations and Schrödinger Equations with Aharonov–Bohm Fields

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## 1 Introduction

It is well known that, if any harmonic function  $u(x)$  in a domain  $\Omega \subset \mathbf{R}^n$  satisfies

$$\partial_x^\alpha u(x_0) = 0$$

for all multi-indices  $\alpha$  at a point  $x_0 \in \Omega$ , then  $u(x)$  vanishes identically in  $\Omega$ . Recently, it is shown by Grammatico [3] that, if  $\Omega$  contains the origin and  $u \in W_{\text{loc}}^{2,2}(\Omega)$  (Sobolev space) satisfies

$$|\Delta u| \leq \frac{M}{|x|^2} |u(x)| + \frac{C}{|x|} |\nabla u| \quad (1)$$

(a.e. on  $\Omega$ ) with  $M > 0$  and  $0 < C < 1/\sqrt{2}$ , and

$$\lim_{\varepsilon \rightarrow +0} \varepsilon^{-\ell} \int_{|x| < \varepsilon} |u|^2 dx = 0, \quad (2)$$

then  $u(x)$  vanishes identically in  $\Omega$  (one can see some related works in the References of Grammatico [3]). Then we say that the inequality (1) has the strong unique continuation property. If  $u(x)$  satisfies (2),  $u(x)$  is said to vanish of infinite order at the origin, or to be flat at the origin. We can not expect the strong unique continuation property for every  $C > 0$ . For Alinhac–Baouendi [1] shows that, if  $C > 1$ , there is a non-trivial complex-valued function  $v \in C^\infty(\mathbf{R}^2)$ , which is flat at the origin satisfying  $\text{supp } v = \mathbf{R}^2$  and (1) with  $M = 0$  (see also Pan–Wolff [7]).

For corresponding problems to the Dirac operator

$$L_0 = \sum_{j=1}^n \alpha_j p_j \quad \left( p_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad n \geq 2 \right),$$

where  $\alpha_j$  are  $N \times N$  Hermitian matrices satisfying  $\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_N$  ( $N = 2^{[(n+1)/2]}$ ), De Carli–Ōkaji [2] shows that, if a positive constant  $C < 1/2$ , then the inequality

$$|L_0 u| \leq \frac{C}{|x|} |u| \quad \text{a.e. on } \Omega \quad (u \in W_{\text{loc}}^{1,2}(\Omega)^N) \quad (3)$$

has the strong unique continuation property, where  $|u| = \sqrt{|u_1|^2 + |u_2|^2}$  (see also Kalf–Yamada [5] and Ōkaji [6]). The restriction on  $C < 1/2$  is needed to treat the angular momentum term (spin–orbit term) but the radial part of  $L_0$ . As is also pointed out by De Carli–Ōkaji [2], the counter example by Alinhac–Baouendi [1] implies that a certain restriction on the constant  $C$  in (3) is also necessary. In fact, if we set

$$u_1 := \partial u = (\partial_1 - i\partial_2)v, \quad u_2 := \bar{\partial} u = (\partial_1 + i\partial_2)v,$$

then we can see that  $u_1$  and  $u_2 \neq 0$  are flat at the origin satisfying (1) with the same constant  $C > 1$  (cf. Corollary below). It is an open problem what happens for  $1/2 \leq C \leq 1$ . In this note we investigate the strong unique continuation property for 2-dimensional Dirac operators with Aharonov–Bohm effect, which is one of singular magnetic fields at the origin, and give a perturbation to the spin–orbit term. Our proof is given along the same line as in De Carli–Ōkaji [2] and Kalf–Yamada [5].

## 2 The Result

Let us consider 2-dimensional Dirac operators with Aharonov–Bohm fields

$$L_\beta := \sigma \cdot D = \sigma_1 D_1 + \sigma_2 D_2,$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$D_j := p_j - b_j(x) = -i \frac{\partial}{\partial x_j} - b_j(x),$$

$$b_1(x) := -\beta \frac{x_2}{|x|^2}, \quad b_2(x) := \beta \frac{x_1}{|x|^2},$$

and  $\beta$  is a real number. Such a magnetic field has a delicate singularity at the origin in spectral theory (see, e.g., Tamura [8]).

Put  $\tilde{\beta} := \beta - [\beta]$ , where  $[\cdot]$  is Gauss’s symbol.

**Theorem 1.** Let  $\Omega$  be a connected open set in  $\mathbf{R}^2$  containing the origin. If  $u \in W_{\text{loc}}^{1,2}(\Omega)^2$  is flat at the origin and

$$|L_\beta u| \leq \frac{C_0}{|x|} |u| \tag{4}$$

a.e. on  $\Omega$  for a positive constant  $C_0 < \gamma(\beta)$  with

$$\gamma(\beta) := \begin{cases} \frac{1-2\tilde{\beta}}{2} & \left(0 \leq \tilde{\beta} < \frac{1}{4}\right), \\ \tilde{\beta} & \left(\frac{1}{4} \leq \tilde{\beta} < \frac{1}{2}\right), \\ 1-\tilde{\beta} & \left(\frac{1}{2} \leq \tilde{\beta} < \frac{3}{4}\right), \\ \frac{2\tilde{\beta}-1}{2} & \left(\frac{3}{4} \leq \tilde{\beta} < 1\right), \end{cases}$$

then  $u$  vanishes identically on  $\Omega$ .

**Corollary.** Let  $S_\beta := D_1^2 + D_2^2$  be the Schrödinger operator. Let  $\Omega$  be an open set containing the origin. If  $v \in W_{\text{loc}}^{2,2}(\Omega)$  is flat at the origin satisfying

$$|S_\beta v| \leq \frac{C_0}{|x|} |Dv| \quad (5)$$

a.e. on  $\Omega$  for a positive constant  $C_0 < \gamma(\beta)$ , then  $v$  vanishes identically on  $\Omega$ , where  $|Dv| := \sqrt{|D_1 v|^2 + |D_2 v|^2}$ .

For the proof of Corollary, let us put  $u_1 := (D_1 - iD_2)v$  and  $u_2 := (D_1 + iD_2)v$ . Since  $v$  is flat at the origin, we can show that  $D_1 v$  and  $D_2 v$  are flat at the origin by using (5). Therefore,  $u_1$  and  $u_2$  are flat at the origin and satisfy

$$\begin{aligned} D_1 v &= \frac{u_1 + u_2}{2}, \quad D_2 v = -\frac{u_1 - u_2}{2i}, \\ D_1 D_2 v &= D_2 D_1 v. \end{aligned}$$

Moreover, we have

$$\begin{aligned} |L_\beta u| &= \sqrt{2} |(D_1^2 + D_2^2)v| \leq \frac{\sqrt{2} C_0}{|x|} |Dv| \\ &= \frac{C_0}{\sqrt{2}|x|} \sqrt{|u_1 - u_2|^2 + |u_1 + u_2|^2} \\ &= \frac{C_0}{|x|} |u|, \end{aligned}$$

which gives from Theorem 1 that  $u_1 = u_2 \equiv 0$  and  $\frac{\partial v}{\partial r} \equiv 0$  in  $\Omega$ . Since  $v$  is flat at the origin, we have  $v \equiv 0$ .

Moreover, applying the proof of Grammatico [3], we can prove the above property even if  $C_0 < \sqrt{2} \gamma(\beta)$ . In fact, we can see the following result:

**Theorem 2.** If  $v \in W_{\text{loc}}^{2,2}(\Omega)$  is flat at the origin satisfying

$$|S_\beta v|^2 \leq \frac{M^2}{|x|^4} |v|^2 + \frac{A^2}{|x|^2} |\partial_r v|^2 + \frac{B^2}{|x|^4} |(\partial_\theta - i\beta)v|^2 \quad (6)$$

a.e. on  $\Omega$ , with positive constants  $M, A, B$  such that  $A^2 + B^2 < 4\gamma(\beta)^2$ , then  $v$  vanishes identically on  $\Omega$ , where  $(r, \theta)$  is the polar coordinate and  $\partial_r = \partial/\partial r$ ,  $\partial_\theta = \partial/\partial \theta$ .

Therefore, if  $v \in W_{\text{loc}}^{2,2}(\Omega)$  is flat at the origin satisfying

$$|S_\beta v| \leq \frac{C_0}{|x|} |Dv|$$

a.e. on  $\Omega$  for a positive constant  $C_0 < \sqrt{2}\gamma(\beta)$ , then  $v$  vanishes identically on  $\Omega$ , by setting  $A = B$  and  $M = 0$  in (6).

### 3 Proof of Theorem 1

Here we introduce some notations. Let

$$D_r := \sum_{j=1}^2 \frac{x_j}{r} D_j, \quad \sigma_r = \sum_{j=1}^2 \frac{x_j}{r} \sigma_j,$$

$$\begin{aligned} S &:= \frac{1}{2} - i\sigma_1\sigma_2(x_1D_2 - x_2D_1) \\ &= \frac{1}{2} + \sigma_3(x_1p_2 - x_2p_1 - \beta), \end{aligned}$$

where

$$\sigma_3 := -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin-orbit operator  $S$  is written by polar coordinates  $x_1 = r \cos \theta$  and  $x_2 = r \sin \theta$  as

$$S = \begin{pmatrix} \frac{1}{2} - \beta - i\frac{\partial}{\partial \theta} & 0 \\ 0 & \frac{1}{2} + \beta + i\frac{\partial}{\partial \theta} \end{pmatrix}, \quad (7)$$

which can be regarded as a self-adjoint operator on  $L^2(S^1)^2$ . Then we have

$$\sigma \cdot D = \sigma_r \left( D_r + \frac{i}{r} S \right), \quad \sigma_r^2 = I, \quad (8)$$

$$\sigma_r D_r = D_r \sigma_r, \quad \sigma_r S = -S \sigma_r, \quad D_r S = S D_r, \quad (9)$$

$$D_r^2 \geq \frac{1}{4r^2} \quad (10)$$

on  $C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ . The last inequality can be shown by a commutator relation

$$\left[ D_r, \frac{1}{r} \right] = \frac{i}{r^2}.$$

**Lemma 2.** For a real number  $m$  we put

$$A := \sigma \cdot D - i \frac{m}{r} \sigma_r.$$

Then we have

$$A^* A \geq \frac{1}{r^2} \left( S - m - \frac{1}{2} \right)^2 \quad (11)$$

on  $C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ , and the spectrum  $\sigma(S)$  consists of discrete eigenvalues

$$\left\{ n + \frac{1}{2} \pm \beta \mid n \in \mathbf{Z} \right\}. \quad (12)$$

*Proof.* The properties (8), (9) and (10) give

$$\begin{aligned} A^* A &= \left[ \sigma_r \left( D_r + \frac{i}{r} S \right) + \frac{im}{r} \sigma_r \right] \\ &\quad \cdot \left[ \sigma_r \left( D_r + \frac{i}{r} S \right) - \frac{im}{r} \sigma_r \right] \\ &= \left[ D_r - \frac{i}{r} (S - m) \right] \left[ D_r + \frac{i}{r} (S - m) \right] \\ &= D_r^2 - \frac{1}{4r^2} + \frac{1}{r^2} \left( S - m - \frac{1}{2} \right)^2 \\ &\geq \frac{1}{r^2} \left( S - m - \frac{1}{2} \right)^2, \end{aligned}$$

which shows (11). Since  $S$  has a complete orthonormal eigenfunctions in  $L^2(S^1)^2$ ,

$$\frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{in\theta} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ e^{-in\theta} \end{pmatrix} \quad (n \in \mathbf{Z}),$$

we obtain (12).

**Lemma 3.** There exists a sequence of positive numbers  $m_j$  ( $j = 1, 2, \dots$ ) with  $m_j \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$\|r^{-m_j}(\sigma \cdot D)u\| \geq \gamma(\beta) \|r^{-m_j-1}u\|$$

for any  $u \in W^{1,2}(\mathbf{R}^2)^2$  whose support does not include a neighborhood of the origin, where  $\gamma(\beta)$  is what is defined in Theorem 1.

*Proof.* Let  $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ . In view of lemma 2 we have

$$\begin{aligned} &\int_{\mathbf{R}^2} r^{-2m} |\sigma \cdot D\varphi|^2 dx \\ &= \int_{\mathbf{R}^2} |A(r^{-m}\varphi)|^2 dx \\ &\geq \min_{n \in \mathbf{Z}} |n \pm \beta - m|^2 \int_{\mathbf{R}^2} r^{-2m-2} |\varphi|^2 dx \end{aligned}$$

for any  $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$  and  $m \in \mathbf{R}$ . Seeing the definition of  $\gamma(\beta)$  in Theorem 1, we can find a sequence  $m_j \rightarrow \infty$  such that

$$\min_{n \in \mathbf{Z}} |n \pm \beta - m_j|^2 = \gamma(\beta).$$

For a given  $u \in W^{1,2}(\mathbf{R}^2)^2$  whose support does not include a neighborhood of the origin, there exists a sequence  $\{\varphi_j\}_{j=1,2,\dots} \subset C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$  such that  $\varphi_j \rightarrow u$  in  $W^{1,2}(\mathbf{R}^2)$  ( $j \rightarrow \infty$ ), which completes the proof.

Lemma 3 yields the following

**Lemma 4.** Suppose that  $u \in W_{\text{loc}}^{1,2}(\Omega)^2$  is flat at the origin with (4). Let  $B_{R_0} := \{x \in \mathbf{R}^2 \mid |x| < R_0\} \subset \Omega$ . For any  $R_1 < R_0$  there exists a positive constant  $C_1 = C_1(R_0, R_1)$  independent of  $m_j$  such that

$$\begin{aligned} & [\gamma(\beta)^2 - C_0^2] \int_{B_{R_1}} r^{-2m_j-2} |u|^2 dx \\ & \leq 2C_0^2 \int_{R_1 < |x| < R_0} r^{-2m_j-2} |u|^2 dx \\ & \quad + C_1 \int_{R_1 < |x| < R_0} r^{-2m_j} |u|^2 dx, \end{aligned} \tag{13}$$

where  $m_j$  is the one given in Lemma 3.

*Proof.* Fix  $0 < R_1 < R_0$  and take  $\delta > 0$  and a smooth function  $\chi_\delta \in C_0^\infty(0, R_0)$  such that

$$\chi_\delta(r) = \begin{cases} 1 & (\delta \leq r \leq R_1) \\ 0 & (r \leq \delta/2) \end{cases}$$

and

$$|\chi'_\delta(r)| \leq \begin{cases} C_2 \delta^{-1} & (\delta/2 \leq r \leq \delta) \\ C_2 & (R_1 \leq r \leq R_0) \end{cases}$$

for a positive constant  $C$ . Then Lemma 3 and the condition (4) yield

$$\begin{aligned} & \gamma(\beta)^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2} |u|^2 dx \\ & \leq \gamma(\beta)^2 \int r^{-2m_j-2} |\chi_\delta u|^2 dx \\ & \leq \int |r^{-2m_j} (\sigma \cdot D)(\chi_\delta u)|^2 dx \\ & \leq 2 \int_{\delta/2 \leq r \leq \delta} r^{-2m_j} [C_2^2 \delta^{-2} + C_0^2 r^{-2}] |u|^2 dx \\ & \quad + C_0^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2} |u|^2 dx \\ & \quad + 2 \int_{R_1 \leq r \leq R_2} r^{-2m_j} [C_2^2 + C_0^2 r^{-2}] |u|^2 dx. \end{aligned} \tag{14}$$

Since  $u$  is flat at the origin, the last three integrals tend to zero if  $\delta \rightarrow 0$ . Therefore we have (13) with  $C_1 = 2C_0^2$ .

*Proof of Theorem 1.* Let  $B_{R_0} \subset \Omega$  and take  $0 < R_2 < R_1 < R_0$ . In view of (13) we have

$$\begin{aligned}
& [\gamma(\beta)^2 - C_0^2] \left( \frac{R_1}{R_2} \right)^{2m_j} \int_{B_{R_2}} \frac{|u|^2}{r^2} dx \\
& \leq [\gamma(\beta)^2 - C_0^2] R_1^{2m_j} \int_{B_{R_1}} r^{-2m_j-2} |u|^2 dx \\
& \leq 2C_0^2 R_1^{2m_j} \int_{R_1 < |x| < R_0} r^{-2m_j-2} |u|^2 dx \\
& \quad + C_1 R_1^{2m_j} \int_{R_1 < |x| < R_0} r^{-2m_j} |u|^2 dx \\
& \leq 2C_0^2 \int_{R_1 < |x| < R_0} \frac{|u|^2}{r^2} dx \\
& \quad + C_1 \int_{R_1 < |x| < R_0} |u|^2 dx.
\end{aligned}$$

Making  $m_j \rightarrow \infty$ , we have  $u \equiv 0$  in  $B_{R_2}$ . Since  $R_1$  and  $R_2$  are arbitrary, we have  $u \equiv 0$  in  $B_{R_0}$ .

Assume that there is  $x_0 \in \Omega$  with  $|x_0| = R_0$ . The condition (3) yields

$$|L_0 u| \leq \frac{C_0 + |\beta|}{|x|} |u| \quad \text{in } \Omega.$$

Set  $x_\varepsilon = (1 - \varepsilon)x_0$  for  $0 < \varepsilon < R_0$ . If

$$0 < \rho < \frac{R_0 - \varepsilon}{1 + 2(C_0 + |\beta|)},$$

then we can find a positive constant  $C' < 1/2$  such that

$$|L_0 u| \leq \frac{C'}{|x - x_\varepsilon|} |u| \quad \text{in } \Omega \cap B_\rho(x_\varepsilon),$$

where  $B_\rho(x_\varepsilon)$  is the open ball with radius  $\rho$  and center  $x_\varepsilon$ . This fact implies, by De Carli-Ökajı [2],

$$u \equiv 0 \quad \text{in } \Omega \cap B_{R_1},$$

where  $R_1 := R_0 [1 + \{2(C_0 + |\beta|) + 1\}^{-1}]$ . By repeating this procedure we have  $u \equiv 0$  in  $\Omega$ .

## 4 Proof of Theorem 2

We shall apply the method developed in Grammatico[3] to (6). The spectrum  $\gamma(\Delta'_\theta)$  coincides of eigenvalues  $\{(k - \beta)^2 \mid k \in \mathbf{Z}\}$  with the corresponding eigenfunction  $\varphi_k(\theta) = (1/\sqrt{2\pi})e^{ik\theta}$ .



We introduce the coordinates  $(T, \theta) \in \mathbf{R} \times S^1$  with  $T = \log r$ .

For  $V \in C_0^\infty(\mathbf{R} \times S^1)$  we write

$$V(T, \theta) = \sum_{k \in \mathbf{Z}} f_k(T) \varphi_k(\theta).$$

We note that

$$\int \int |V(T, \theta)|^2 dT d\theta = \sum_{k \in \mathbf{Z}} \int |f_k(T)|^2 dT,$$

since

$$\|V(T, \cdot)\|_{L^2(S^1)}^2 = \sum_{k \in \mathbf{Z}} |f_k(T)|^2,$$

where  $\|\cdot\|$  denotes the  $L^2(S^1)$ -norm. Set

$$Q = r^2 S_\beta$$

and

$$Q_\tau = e^{-\tau T} (Q e^{\tau T} V),$$

where  $\tau$  is a real parameter.

We can see directly

$$Q_\tau V = -(\partial_T^2 + 2\tau \partial_T + \tau^2 + \Delta'_\theta) V.$$

Hence we have

$$\begin{aligned} \int \|Q_\tau V(T, \cdot)\|^2 dT &= \int \|\partial_T^2 V(T, \cdot)\|^2 + 2 \int \langle \partial_T^2 V, \Delta'_\theta V \rangle dT + 2\tau^2 \int \|\partial_T V(T, \cdot)\|^2 dT \\ &\quad + \tau^4 \int \|V(T, \cdot)\|^2 dT + 2\tau^2 \int \langle V, \Delta'_\theta V \rangle dT + \int \|\Delta'_\theta V(T, \cdot)\|^2 dT. \end{aligned}$$

Since we obtain

$$\int \langle \partial_T^2 V, \Delta'_\theta V \rangle dT = \int dT \int |\partial_T \Omega_\beta V|^2 d\theta \geq 0$$

by using  $\Delta'_\theta = \Omega_\beta^* \Omega_\beta$ , we have

$$\begin{aligned} \int \|Q_\tau V(T, \cdot)\|^2 dT &\geq 2\tau^2 \int \|\partial_T V(T, \cdot)\|^2 dT + \tau^4 \int \|V(T, \cdot)\|^2 dT \\ &\quad + 2\tau^2 \int \langle V, \Delta'_\theta V \rangle dT + \int \|\Delta'_\theta V(T, \cdot)\|^2 dT \end{aligned}$$

and consequently

$$\begin{aligned} \int \|Q_\tau V(T, \cdot)\|^2 dT &\geq \tau^4 \sum_{k \in \mathbf{Z}} \int |f_k(T)|^2 dT - 2\tau^2 \sum_{k \in \mathbf{Z}} (k - \beta)^2 \int |f_k(T)|^2 dT \\ &\quad + \sum_{k \in \mathbf{Z}} (k - \beta)^4 \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T V(T, \cdot)\|^2 dT. \end{aligned}$$

The later inequality can be written as

$$\int \|Q_\tau V(T, \cdot)\|^2 dT \geq \sum_{k \in \mathbf{Z}} (\tau^2 - (k - \beta)^2)^2 \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T v(T, \cdot)\|^2 dT. \quad (15)$$

Seeing the definition of  $\gamma(\beta)$  in Theorem 1, we can find a sequence  $\tau_j \rightarrow \infty$  such that

$$\min_{k \in \mathbf{Z}} \frac{(\tau_j^2 - (k - \beta)^2)^2}{(k - \beta)^2} = C_\beta, \quad (16)$$

where  $C_\beta = 4\gamma(\beta)^2$ . Then we obtain from (15)

$$\int \|Q_\tau V(T, \cdot)\|^2 dT \geq C_\beta \sum_{k \in \mathbf{Z}} (k - \beta)^2 \int |f_k(T)|^2 dT = C_\beta \int \|\Omega_\beta V(T, \cdot)\|^2 dT. \quad (17)$$

Setting  $U = e^{\tau T} V$ , the above inequality can be written as

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT. \quad (18)$$

For any  $C'_\beta < C_\beta$  we can find a sufficiently large  $\tau_0$  such that

$$(\tau^2 - (k - \beta)^2)^2 \geq C'_\beta \geq \tau^2$$

for any  $\tau \geq \tau_0$  satisfying (16). Then, in view of (15) we have

$$\begin{aligned} \int \|Q_\tau V(T, \cdot)\|^2 dT &\geq C'_\beta \tau^2 \sum_{k \in \mathbf{Z}} \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T V(T, \cdot)\|^2 dT \\ &\geq C'_\beta \left( \tau^2 \int \|V(T, \cdot)\|^2 dT + \int \|\partial_T V(T, \cdot)\|^2 dT \right). \end{aligned} \quad (19)$$

From now on, we consider  $\tau \geq \tau_0$  satisfying (16). We recall  $U = e^{\tau T} V$  so that

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT. \quad (20)$$

For any  $\alpha \in [0, 1]$  we have from (18) and (20)

$$\begin{aligned} \int e^{-2\tau T} \|QU\|^2 dT &\geq \alpha C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT \\ &\quad + (1 - \alpha) C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT. \end{aligned} \quad (21)$$

On the other hand, we obtain from (19)

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C'_\beta \tau^2 \int e^{-2\tau T} \|U\|^2 dT. \quad (22)$$

Therefore, (21) and (22) give

$$\begin{aligned} \left(1 + \frac{1}{\tau'}\right) \int e^{-2\tau T} \|QU\|^2 dT &\geq \tau' \int e^{-2\tau T} \|U\|^2 dT + \alpha C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT \\ &\quad + (1 - \alpha) C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT, \end{aligned} \quad (23)$$

where  $\tau' = C'_\beta{}^{\frac{1}{2}} \tau$ .

Now let us set  $W_c^{2,2}(\mathbf{R}^2) = \{v \mid v \in W^{2,2}(\mathbf{R}^2) \text{ has a compact support}\}$ . The Inequality (23) still holds for  $v \in W_c^{2,2}(\mathbf{R}^2)$ , since the fact follows from the denseness of  $C_0^\infty(\mathbf{R}^2)$  in  $W_c^{2,2}(\mathbf{R}^2)$ .

We set  $B_R = \{x \in \mathbf{R}^2 \mid |x| < R\} \subset \Omega$  and choose  $\chi(T) \in C^\infty(\mathbf{R})$  such that  $0 \leq \chi \leq 1$  and

$$\chi(T) = \begin{cases} 1, & T < T_0 \\ 0, & T > \log R, \end{cases}$$

where  $e^{T_0} < R$ . Let  $\phi \in C^\infty(\mathbf{R}^2)$  such that

$$\phi(T) = \begin{cases} 0, & |x| < \frac{1}{2} \\ 1, & |x| > 1. \end{cases}$$

and  $\phi_j(x) = \phi(jx)$  ( $j \in \mathbf{N}$ ).

Let  $u \in W_{loc}^{2,2}(\mathbf{R}^2)$  be flat at the origin for which (6) holds. Then the functions  $\phi_j \chi u \in W_c^{2,2}(\mathbf{R}^2)$  satisfy (23). If we take the limit as  $j \rightarrow \infty$ , we see that  $\chi u$  also satisfies (23).

By  $(T, \theta)$  coordinates (6) becomes

$$|Qu|^2 = |e^{2\tau T} S_\beta u|^2 \leq M^2 |u|^2 + A^2 |\partial_T u|^2 + B^2 |\Omega_\beta u|^2 \quad (24)$$

for  $T < \log R$ .

By applying (23) to  $\chi u$  we have for  $\tau$  big enough

$$\begin{aligned} (1 + \frac{1}{\tau'}) & \left( \int_{-\infty}^{T_0} e^{-2\tau T} \|Qu(T, \cdot)\|^2 dT + \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T, \cdot)\|^2 dT \right) \\ & \geq \tau' \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T, \cdot)\|^2 dT + \alpha C'_\beta \int_{-\infty}^{T_0} e^{-2\tau T} |\partial_T u(T, \cdot)|^2 dT \\ & \quad + (1 - \alpha) C_\beta \int_{-\infty}^{T_0} e^{-2\tau T} \|\Omega_\beta u(T, \cdot)\|^2 dT. \end{aligned} \quad (25)$$

If we set

$$\psi(T) = M^2 \|u(T, \cdot)\|^2 + A^2 \|\partial_T u(T, \cdot)\|^2 + B^2 \|\Omega_\beta u(T, \cdot)\|^2,$$

then we obtain from (24) and (25)

$$\begin{aligned} (1 + \frac{1}{\tau'}) & \left( \int_{-\infty}^{T_0} e^{-2\tau T} \psi(T) dT + \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T, \cdot)\|^2 dT \right) \\ & \geq \tau' \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T, \cdot)\|^2 dT + \alpha C'_\beta \int_{-\infty}^{T_0} e^{-2\tau T} |\partial_T u(T, \cdot)|^2 dT \\ & \quad + (1 - \alpha) C_\beta \int_{-\infty}^{T_0} e^{-2\tau T} \|\Omega_\beta u(T, \cdot)\|^2 dT, \end{aligned}$$

that is,

$$\begin{aligned}
& \left(1 + \frac{1}{\tau'}\right) \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T, \cdot)\|^2 dT \\
& \geq \left(\tau' - M^2 \left(1 + \frac{1}{\tau'}\right)\right) \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T, \cdot)\|^2 dT \\
& \quad + \left(\alpha C'_\beta - A^2 \left(1 + \frac{1}{\tau'}\right)\right) \int_{-\infty}^{T_0} e^{-2\tau T} |\partial_T u(T, \cdot)|^2 dT \\
& \quad + \left((1 - \alpha) C_\beta - B^2 \left(1 + \frac{1}{\tau'}\right)\right) \int_{-\infty}^{T_0} e^{-2\tau T} \|\Omega_\beta u(T, \cdot)\|^2 dT.
\end{aligned}$$

Now, if  $A^2 + B^2 < C_\beta$  and  $\tau$  is big enough, we can choose any  $C'_\beta < C_\beta$  and  $\alpha \in [0, 1]$  such that

$$\alpha C'_\beta - A^2 \left(1 + \left(\frac{1}{\tau'}\right)\right) > 0, \quad (1 - \alpha) C_\beta - B^2 \left(1 + \left(\frac{1}{\tau'}\right)\right) > 0.$$

Thus, we have

$$\begin{aligned}
& e^{-2\tau T_0} \left(1 + \frac{1}{\tau'}\right) \int_{T_0}^{+\infty} \|Q(\chi u)(T, \cdot)\|^2 dT \\
& \geq \left(1 + \frac{1}{\tau'}\right) \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T, \cdot)\|^2 dT \\
& \geq \left(\tau' - M^2 \left(1 + \frac{1}{\tau'}\right)\right) \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T, \cdot)\|^2 dT \\
& \geq e^{-2\tau T_0} \left(\tau' - M^2 \left(1 + \frac{1}{\tau'}\right)\right) \int_{-\infty}^{T_0} \|u(T, \cdot)\|^2 dT.
\end{aligned}$$

Making  $\tau = \tau_j \rightarrow \infty$ , we have  $u \equiv 0$  in  $\{x \in \mathbf{R}^2 \mid |x| < e^{T_0}\}$ , and therefore  $u \equiv 0$  in  $B_R$ . With the similar argument in Theorem 1, we have  $u \equiv 0$  in  $\Omega$ .

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